

# Unified Approach to Crossover Phenomena

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A general analytical method is developed for describing crossover phenomena of arbitrary nature. The method is based on the algebraic self-similar renormalization of asymptotic series, with control functions defined by crossover conditions. The method can be employed for such difficult problems for which only a few terms of asymptotic expansions are available, and no other techniques are applicable. As an illustration, analytical solutions for several important physical problems are presented.

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## I. INTRODUCTION

Crossover phenomena are ubiquitous in nature. Probably, they are much more common than phase transitions. When speaking about crossover phenomena, one usually keeps in mind the following picture: A function  $f(x)$ , describing a physical quantity, is continuous in an interval  $x_1 \leq x \leq x_2$ , but the behaviour of  $f(x)$  in the vicinity of the boundaries of this interval is *qualitatively different* near  $x_1$  as compared to  $x_2$ . The qualitative change of the behaviour of this function, as  $x$  moves from one side to another side is commonly understood as a crossover.

It is possible to quote hundreds of examples of different crossovers. For instance, many physical quantities qualitatively change their behaviour when passing from the weak-coupling to strong-coupling limit [1]. This concerns, e.g., the majority of problems having to do with the behaviour of energies as functions of a coupling parameter in statistical physics, quantum mechanics, and field theory. Let us mention in this respect the dependence of the spectra of Schrödinger operators on the anharmonicity parameter for variegated anharmonic models. The energy spectrum of such models is qualitatively different in the weak-coupling (weak anharmonicity) as compared to the strong-coupling (strong anharmonicity) limits.

A famous example of a crossover phenomenon is the Kondo effect [2] when the behaviour of a system changes qualitatively at varying temperature. Although this transformation goes smoothly, with no discontinuities in thermodynamic characteristics, but the change of properties is so noticeable that one can ascribe a particular point, called the Kondo temperature, to a region dividing qualitatively different regimes of low and high temperatures.

Another renowned example of a crossover is the Fröhlich polaron problem [3]. Polaron characteristics, such as its energy or effective mass, change qualitatively when varying the coupling parameter describing electron-phonon interactions. This change happens so explicitly that for about two decades there were many speculations suggesting that there exists a phase transition at a particular value of the coupling parameter. However, modern highly accurate Monte Carlo calculations [4] confirm the initial Feynman picture [3] proving that we meet here not a phase transition but a classical crossover.

In the examples mentioned above, of simple anharmonic models, the Kondo effect, and of the Fröhlich polaron problem, the crossover, when varying a coupling parameter or temperature, is monotonic. However, there are cases when crossover is not monotonic. This concerns, for instance, one-dimensional antiferromagnet whose characteristics are considered as functions of spin. Then the Haldane gap [5], as spin changes from small to large values exhibits a very nonmonotonic behaviour becoming zero at each half-odd-integer spin.

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A nonmonotonic behaviour can often be met in the dependence of spectra of collective excitations on wave vector. Then the crossover from the region of small wave numbers, corresponding to the long-wavelength acoustic regime, to the region of large wave vectors, corresponding to a single-particle regime, can go through a nontrivial intermediate region displaying maxima and minima, associated with maxons and rotons [6-8].

We could adduce a number of other examples of crossover phenomena related to interesting physical problems. Let us just mention deconfinement in nuclear matter, which is rather a crossover phenomenon than a phase transition (see discussion in review [9]). However, we think, it is already clear that crossover phenomena are widespread in nature and that it is important to know how to describe them.

The description of crossover phenomena occurring in realistic statistical systems is usually very complicated. This is because one needs to find physical characteristics for a wide range of parameters, which is far from being trivial for complex systems. Say, we have to find a functions  $f(x)$  on the whole semi-axis  $0 \leq x < \infty$ . The variable  $x$  may represent, e.g., a coupling parameter, temperature, or wave vector. Quite often, one can define, more or less easily, the asymptotic behaviour of  $f(x)$  near the boundaries of the interval  $[0, \infty)$ , that is, when  $x \rightarrow 0$  and  $x \rightarrow \infty$ . Such asymptotic expressions may correspond to the weak-coupling and strong coupling limits, to the low-temperature and high-temperature expansions, to the long-wavelength and short-wavelength approximations, and so on. But in the intermediate region, where there are no small parameters, one cannot invoke perturbative techniques. It would be nice to possess a method allowing to construct interpolation formulas only from the knowledge of asymptotic expansions near boundaries.

There exist several summation techniques, such as Padé approximation, Borel summation, conformal mapping, and so on, that permit to ascribe effective sums to asymptotic series [1,10]. But all these techniques are not applicable in principle for the complex problems we are interested in here. This is because of the following main reasons: First, all these summation techniques, to be applicable, require the knowledge of tens of terms in an asymptotic series. Such a luxurious information is usually not available for nontrivial systems, for which standardly one is able to derive just a few perturbative terms. Then all mentioned summation techniques are useless. Second, the latter are just *summation* methods, while we here are concerned about an *interpolation* problem. Summation and interpolation are far from being the same!

The most known interpolation method is the *two-point* Padé approximation [11], that should not be confused with the standard Padé approximation [10]. However, the former method, being a derivative of the latter, shares all its deficiencies. Among the most important shortcomings of Padé approximants, we may mention the following: the necessity of having many perturbative terms, the appearance of unphysical poles, the ability of dealing only with the so-called compatible variables, the possibility of describing only those functions that have at infinity a power-law behaviour with rational powers, and the impossibility to correctly treat nonmonotonic crossover. These difficulties are well known and repeatedly discussed in literature [10-18]. In addition, we remind that Padé approximation is rather a numerical method.

In the present paper, we advance an *analytical* approach for treating interpolation problems of arbitrary nature. This approach is free of the shortcomings typical of the two-point Padé approximation. And what also makes this approach more general than any other know methods is the possibility of using it for those difficult cases when just a few asymptotic terms are available and no other method is applicable. We illustrate the approach by applying it to several difficult problems with monotonic as well as with nonmonotonic crossover. We would like to stress that the physical problems we consider not only illustrate the wide applicability of the suggested approach but are also of interest as such. Therefore, the interpolation formulas we derive present analytical solutions for important physical problems.

## II. GENERAL APPROACH

The interpolation approach we advance here is based on the ideas of our previous papers. However, since we do not assume that a reader in the common audience is already well acquainted with these ideas, we provide here a clearly understandable description of the method in general, complementing it by those particulars that are necessary for adjusting it to the interpolation problem

Assume that we are looking for a physical characteristic presented by a function  $f(x)$ , in which the variable  $x$  changes in the interval  $[0, \infty)$ . The standard situation is when the physical problem under investigation is so complicated that it is difficult, or even impossible, to find a reasonable approximation for the sought function in the whole given interval. However, it is often feasible to get an asymptotic expansion for small variables

$$f(x) \simeq p_k(x), \quad (x \rightarrow 0), \quad (1)$$

where  $k = 0, 1, 2, \dots$ , employing a kind of perturbation theory. Also, it is often possible to find an asymptotic behaviour of the function at large variables, say,

$$f(x) \simeq f_{as}(x) \quad (x \rightarrow \infty). \quad (2)$$

Then the interpolation problem consists in answering the question: What can be said about the behaviour of the function in the whole interval  $[0, \infty)$  being based on the asymptotic information in (1) and (2)? Usually, not much, since the asymptotic expressions (1) and (2), being derived in two opposite limits, have nothing common with each other. In addition, perturbative approximations, such as (1), usually result in divergent series. When one is lucky enough, dealing with a more or less easy case, so that tens of perturbative terms in (1) could be calculated, then one could invoke some known summation technique in order to ascribe an effective sum to a divergent series. However, even in such a lucky case, the found effective sum may have, and usually has nothing to do with the limit (2) from another side of the axis  $[0, \infty)$ . For example, the standard case is when the limit (2) corresponds to an exponential behaviour. If one uses Padé approximants or any other techniques based on them, for instance, Padé-Borel summation [1], one comes to an effective sum in the form of rational fractions, which cannot be matched with an exponential. In the less lucky but more realistic case, when only a few perturbative terms are known, all these summation techniques become in principle useless. How could we proceed in such difficult cases in order to find an interpolation formula connecting (1) and (2)?

The first thing we need to do is to understand how to extract a useful information from a divergent sequence  $\{p_k(x)\}$  when only a few initial terms of it are available. It would be nice to reconstruct the sequence  $\{p_k(x)\}$  in such a way that to improve its convergence properties. Having only a few terms, we cannot resort to the standard summation techniques. Nevertheless, a reconstruction is possible with the help of *control functions* [19,20]. Let us denote the procedure of introducing control functions as

$$C_s\{p_k(x)\} = P_k(x, s), \quad (3)$$

where  $s = s_k(x)$  is a set of functions such that the sequence  $\{P_k(x, s_k)\}$  has better convergence properties than  $\{p_k(x)\}$ . The name "control functions" reflects their role in controlling convergence. The introduction of such functions can be done in several ways. Generally, any procedure of obtaining a sequence of approximations consists of three elements: of a calculational algorithm, an initial approximation, and of additional transformations. For example, by introducing a relaxation or damping parameter into the numerical Newton method, one can improve the convergence of the latter [21]. Under a given calculational algorithm, one may include control functions into an initial approximation  $P_0(x, s)$ , after which all following approximations also become dependent on  $s$ . This variant of introducing control functions is, probably, the most widely used. One takes an initial Hamiltonian, or Lagrangian, or action as depending on trial parameters that are defined as control functions by imposing an addition condition, like the minimal-difference condition [19,20,22-26] or the minimal-sensitivity condition [27-32]. Such conditions are, of course, heuristic. For simple cases, as a zero-dimensional and one-dimensional oscillators, for which perturbative terms of arbitrary order can be obtained, one may define control functions directly from the condition of convergence of these terms, as  $k \rightarrow \infty$  [33-35]. Finally, if a calculational algorithm with an initial approximation have been fixed, one may introduce control functions by subjecting the resulting asymptotic series to additional transformations. These can be either a change of variables, with the reexpansion of the given series in powers of new variables, or a transformation of a series itself. An example of the former case is the order dependent mapping, and that of the latter, the Borel-Leroy transformation [1]. However, these transformations require the knowledge of the analyticity properties of the sought functions itself, which is rarely available.

To our mind, a transformation that one wishes to apply to an asymptotic series in order to construct an analytical approach must satisfy three main stipulations: (i) Be *general*, to be applicable to any function without requiring the knowledge of its properties that are not known. The sole assumption involved should be the existence of the sought function. (ii) Be *simple*, to permit an analytical investigation. At the same time, simplicity is usually a requisit for generality. (iii) Be *invertible*, with a uniquely defined inverse transformation. This is evidently necessary to return from a transform to the function itself. In addition, it would be desirable to have an apparent interpretation of the meaning of the chosen transformation.

These stipulations are satisfied by the algebraic transformation [36-38] whose general form is

$$C_s\{f(x)\} = a(x, s) + b(x, s)f(x), \quad (4)$$

where  $a(x, s)$  and  $b(x, s)$  are any functions guarantying the uniqueness of the inverse transformation

$$C_s^{-1}\{C_s\{f(x)\}\} = \frac{C_s\{f(x)\} - a(x, s)}{b(x, s)} = f(x).$$

One of the simplest variants of (4), as applied to a term  $p_k(x)$  of a sequence  $\{p_k(x)\}$ , is

$$C_s\{p_k(x)\} = P_k(x, s) = x^s p_k(x). \quad (5)$$

This variant not only simple but also it has a transparent meaning when  $p_k(x)$  is a  $k$ -order truncated series in powers of  $x$ . Then, transformation (5) effectively increases the approximation order from  $k$  to  $k + s$ .

Assume that, by this or that way, we have introduced control functions constructing from an initial sequence  $\{p_k(x)\}$  a transformed sequence  $\{P_k(x, s)\}$  with better convergence properties. Now we have to concretize in what sense the properties of  $\{P_k(x, s)\}$  should be better than those of  $\{p_k(x)\}$ . The greatest achievement would be if the transformed sequence  $\{P_k\}$  is such that we could notice a relation between subsequent terms  $P_k$  and  $P_{k+1}$ . If so, we would be able to map the low-order terms to those of arbitrary high order. That is, having just a few initial terms of a sequence  $\{P_k\}$ , we could extrapolate them to higher orders of  $k$  defining an effective limit  $P^*$  of this sequence. To formulate a relation between subsequent terms of a sequence of approximations means to define the property of self-similarity between these terms. This can be called the *approximation self-similarity* [39-43]. To formulate the latter, we need to invoke some further transformations. To this end, let us define an expansion function  $x(\varphi, s)$  by the equation

$$P_0(x, s) = \varphi, \quad x = x(\varphi, s). \quad (6)$$

Then, we introduce an endomorphism

$$y_k(\varphi, s) \equiv P_k(x(\varphi, s), s), \quad (7)$$

with an initial condition

$$y_0(\varphi, s) \equiv \varphi, \quad (8)$$

following from (6). The transformation inverse to (7) reads

$$P_k(x, s) = y_k(P_0(x, s), s). \quad (9)$$

By these definitions, the sequence  $\{y_k(\varphi, s)\}$  is bijective to  $\{P_k(x, s)\}$ . The property of self-similarity between the terms of the sequence  $\{y_k(\varphi, s)\}$  writes [39-43] as

$$y_{k+p}(\varphi, s) = y_k(y_p(\varphi, s), s). \quad (10)$$

This is nothing but the semigroup property  $y_{k+p} = y_k \cdot y_p$ . Relation (10) may remind a functional equation of renormalization group [1,44]. However, there is here a principal difference. Renormalization-group equations [1,44] relate a function with scaled variables with the function itself. So, a renormalization-group equation describes motion with respect to function variables. In our case, Eq. (10) relates different approximations from the sequence  $\{y_k\}$ . Therefore, the approximation self-similarity (10) defines motion with respect to approximation orders which play the role of discrete time. In the language of dynamical theory, a dynamical system with discrete time is called a cascade. Since the trajectory  $\{y_k(\varphi, s)\}$  of this cascade is, by construction, bijective to the sequence of approximations  $\{P_k(x, s)\}$ , a family of endomorphisms  $\{y_k | k = 0, 1, 2, \dots\}$  can be named the *approximation cascade* [45,46]. An important feature of this cascade is that the approximation self-similarity (10) is a necessary condition for fastest convergence [42,43].

For the purpose of developing an analytical theory, it is not convenient to deal with discrete time. It would be desirable to pass from the discrete index  $k = 0, 1, 2, \dots$  to a continuous variable  $t \in [0, \infty)$ . This can be done [39-42] by introducing an endomorphism  $y_t(\varphi, s)$  such that  $y_t$  has the same group property,

$$y_{t+\tau}(\varphi, s) = y_t(y_\tau(\varphi, s), s), \quad (11)$$

as  $y_k$  in (10), and the values

$$y_t(\varphi, s) = y_k(\varphi, s) \quad (t = k) \quad (12)$$

at integer  $t$  coincide. The so defined family of endomorphisms  $\{y_t | t \in [0, \infty)\}$  forms an *approximation flow*, and conditions (11) and (12) define the *embedding* of a cascade into flow [45,46]. From relation (11) with continuous time, it is easy to derive the Lie evolution equation

$$\frac{\partial}{\partial t} y_t(\varphi, s) = v_t(y_t, s), \quad (13)$$

with the velocity field

$$v_t(y_t, s) = \lim_{\varphi \rightarrow y_t} \lim_{t \rightarrow 0} \frac{\partial}{\partial t} y_t(\varphi, s). \quad (14)$$

Equation (13) can be rewritten in the integral form

$$\int_{y_t}^{y_{t+\tau}} \frac{d\varphi}{v_t(\varphi, s)} = \tau. \quad (15)$$

To study the properties of an approximation flow, we may invoke powerful techniques of dynamical theory [47-50].

What we need to obtain at the end is an effective limit of the sequence  $\{P_k(x, s)\}$ . Since the latter is bijective to the trajectory  $\{y_k(\varphi, s)\}$  of the approximation cascade, the limit of  $\{P_k\}$  is in one-to-one correspondence with a stable fixed point of the cascade [45,46]. A fixed point is defined as a zero of velocity. The cascade velocity can be written as the Euler discretization of the flow velocity [42,43] which reads

$$v_k(\varphi, s) = y_{k+1}(\varphi, s) - y_k(\varphi, s) + \Delta s \frac{\partial}{\partial s} y_k(\varphi, s), \quad (16)$$

where  $\Delta s$  is a variation of a control function. Since this variation is not known, we cannot find an exact zero of the velocity (16), but can find only its approximate zero defining a quasifixed point. For instance, we may put

$$\Delta s \frac{\partial}{\partial s} y_k(\varphi, s) = 0, \quad (17)$$

which is satisfied if either  $\Delta s = 0$  or  $\partial y_k / \partial s = 0$ . In both the cases, the velocity (16) becomes

$$v_k(\varphi, s) = y_{k+1}(\varphi, s) - y_k(\varphi, s). \quad (18)$$

This and several other ways of defining quasifixed points and the related velocities have been analysed in detail in Refs. [51-53]. The motion in the space of approximations, near a quasifixed point, is described by the evolution integral (15), which can be written as

$$\int_{P_k}^{P_{k+1}^*} \frac{d\varphi}{v_k(\varphi, s)} = \tau, \quad (19)$$

where  $P_k = P_k(x, s)$ ;  $P_k^* = P_k^*(x, s, \tau)$  is a quasifixed point; and  $\tau$  is a minimal time necessary to reach this quasifixed point.

Substituting the cascade velocity (18) into the evolution integral (19), we can find a quasifixed point  $P_k^*$ . Then, we need to make a transformation inverse to the algebraic transformation (4),

$$p_k^*(x, s, \tau) = C_s^{-1} \{P_k^*(x, s, \tau)\}. \quad (20)$$

The resulting approximant (20) is, as is clear from (19), a function of  $P_{k-1}$ , that is of  $p_{k-1}$ , which can be written as

$$p_k^* = F_k(p_{k-1}).$$

We may repeat the renormalization procedure for  $p_{k-1}$ , obtaining

$$p_k^* = F_k(F_{k-1}(p_{k-2})),$$

and so on. After  $k$  steps of such a procedure, called self-similar bootstrap [38], we come to

$$p_k^* = F_k(F_{k-1}(\dots F_1(p_0))\dots).$$

In short notation, the latter can be presented as a quasifixed-point equation

$$p_k^* = F_k(p_{k-1}^*). \quad (21)$$

As far as (21) implies a  $k$ -step renormalization, the resulting  $p_k^*$  will contain two sets

$$\bar{s}_k = \{s_1, s_2, \dots, s_k\}, \quad \bar{\tau}_k = \{\tau_1, \tau_2, \dots, \tau_k\}$$

of  $2k$  control functions, which can be denoted as

$$p_k^* = p_k^*(x, \bar{s}_k, \bar{\tau}_k).$$

Now it is time to recall the main aim of the present paper, that is, to suggest an approach for treating crossover phenomena. Therefore, we must remember the asymptotic condition (2) and to require that the found approximation (21) would satisfy the condition

$$p_k^*(x, \bar{s}_k, \bar{\tau}_k) \simeq f_{as}(x) \quad (x \rightarrow \infty). \quad (22)$$

This defines the sets  $\bar{s}_k = \bar{s}_k(x)$  and  $\bar{\tau}_k = \bar{\tau}_k(x)$  of control functions. With the found control functions, we obtain the final self-similar approximant

$$f_k^*(x) = p_k^*(x, \bar{s}_k(x), \bar{\tau}_k(x)). \quad (23)$$

What makes the present paper different from our previous publications is the *systematic use of the asymptotic conditions* of type (22) for defining control functions. The suggested procedure is designed so that to self-similarly sew the left and right asymptotic expansions of a function on a given interval. For concreteness, we have considered above the sewing procedure from the left to the right. But, as is evident, the same way can be followed from the right to the left, that is, starting from an asymptotic expansion at the right boundary of the interval  $[0, \infty)$ , when  $x \rightarrow \infty$ , and then sewing the obtained approximant with the asymptotic form at the left boundary, where  $x \rightarrow 0$ . In any case, we shall come to an approximant whose structure is governed by the quasifixed-point equation (21).

To show explicitly what is the structure of the approximant  $p_k^*$ , let us take an initial expansion  $p_k(x)$ , as  $x \rightarrow 0$ , in the form of a standard power series

$$p_k(x) = \sum_{n=0}^k a_n x^n. \quad (24)$$

Employing the algebraic transform (5), we have

$$P_k(x, s) = \sum_{n=0}^k a_n x^{n+s}. \quad (25)$$

Equation (6) reads

$$P_o(x, s) = a_0 x^s = \varphi, \quad (26)$$

from where the expansion function is

$$x = \left( \frac{\varphi}{a_0} \right)^{1/s}. \quad (27)$$

For the endomorphism (7), we get

$$y_k(\varphi, s) = \sum_{n=0}^k a_n \left( \frac{\varphi}{a_0} \right)^{1+n/s}. \quad (28)$$

The cascade velocity (18) becomes

$$v_k(\varphi, s) = a_{k+1} \left( \frac{\varphi}{a_0} \right)^{1+(k+1)/s}. \quad (29)$$

Substituting (29) into the evolution integral (19), we find a quasifixed point  $P_k^*$ , after which we need to make the inverse transformation (20),

$$p_k^*(x, s, \tau) = x^{-s} P_k^*(x, s, \tau). \quad (30)$$

This results in the expression

$$p_k^*(x, s, \tau) = \left[ p_{k-1}^{-k/s}(x) - \frac{k a_k \tau}{s a_0^{1+k/s}} x^k \right]^{-s/k}. \quad (31)$$

Let us note that when  $s \rightarrow \infty$ , then

$$\lim_{s \rightarrow \infty} p_k^*(x, s, \tau) = p_{k-1}(x) \exp\left(\frac{a_k}{a_0} \tau x^k\right), \quad (32)$$

which explains how naturally exponentials appear in our approach [38].

The quasifixed-point equation (21), as applied to (31), gives

$$p_k^* = \left[ (p_{k-1}^*)^{1/n_k} + B_k x^k \right]^{n_k}, \quad (33)$$

where, for brevity, the arguments of  $p_k^* = p_k^*(x, s_k, \tau_k)$  are not written down, and the notation

$$n_k \equiv -\frac{s_k}{k}, \quad B_k \equiv \frac{a_k \tau_k}{n_k a_0^{1-1/n_k}} \quad (34)$$

is used. In the same way, we get

$$p_{k-1}^* = \left[ (p_{k-2}^*)^{1/n_{k-1}} + B_{k-1} x^{k-1} \right]^{n_{k-1}},$$

and so on down to

$$p_2^* = \left[ (p_1^*)^{1/n_2} + B_2 x^2 \right]^{n_2}$$

and

$$p_1^* = \left( p_0^{1/n_1} + B_1 x \right)^{n_1},$$

where  $p_0 = a_0$ . This shows that the structure of (33) is a sequence of nested roots. For instance, a third-order approximant looks like

$$p_3^* = \left\{ \left[ \left( p_0^{1/n_1} + B_1 x \right)^{n_1/n_2} + B_2 x^2 \right]^{n_2/n_3} + B_3 x^3 \right\}^{n_3}.$$

Control functions  $s_k$  and  $\tau_k$  are to be found from the asymptotic condition (22). Because of relation (34), this is the same as to define the powers  $n_k$  and amplitudes  $B_k$ . Equalities (34) are nothing but a change of variables, so that instead of  $s_k$  and  $\tau_k$  we may consider  $n_k$  and  $B_k$  as new control functions. For practical purpose, we may at once write down a  $k$ -order approximant in the form of (33) and to directly define  $n_k$  and  $B_k$  from condition (22). If the latter gives several solutions for control functions, then, we should opt for that solution which leads to the decrease of  $B_k$  with increasing  $k$ . This follows from (33), from where it is evident that  $p_k^*$  tends to a fixed point  $p^*$  if and only if  $B_k \rightarrow 0$ , as  $k \rightarrow \infty$ .

Before passing to the consideration of complex physical problems using the method we have described, let us illustrate it by a simple example of the one-dimensional quartic anharmonic oscillator. Consider the dimensionless ground-state energy  $e(g)$  as a function of the coupling, or anharmonicity, parameter  $g \in [0, \infty)$ . In the weak-coupling limit, when  $g \rightarrow 0$ , perturbation theory results [54] in the expansion

$$e(g) \simeq a_0 + a_1 g + a_2 g^2 + a_3 g^3 + a_4 g^4, \quad (35)$$

where

$$a_0 = \frac{1}{2}, \quad a_1 = \frac{3}{4}, \quad a_2 = -\frac{21}{8}, \quad a_3 = \frac{333}{16}, \quad a_4 = -\frac{30885}{128}.$$

In the strong-coupling limit, when  $g \rightarrow \infty$ , the asymptotic behavior is known [55,56] to be

$$e(g) \simeq A_0 g^{1/3} + A_1 g^{-1/3} + A_2 g^{-1} + A_3 g^{-5/3}, \quad (36)$$

with the coefficients

$$A_0 = 0.667986, \quad A_1 = 0.143669, \quad A_2 = -0.008628, \quad A_3 = 0.000818.$$

Starting from the linear approximation  $p_1(g) = a_0 + a_1(g)$  from (35), we find

$$p_1^*(g, n_1, B_1) = \left( a_0^{1/n_1} + B_1 g \right)^{n_1},$$

which is the first approximation from (33). Requiring the validity of the asymptotic condition

$$p_1^*(g, n_1, B_1) \simeq A_0 g^{1/3} \quad (g \rightarrow \infty),$$

we find  $n_1 = \frac{1}{3}$ ,  $B_1 = A_0^3 = 0.298059$ . From here, we could recalculate  $s_k$  and  $\tau_k$  using relations (34), however, this is not necessary, since, as is explained above, now  $n_k$  and  $B_k$  play the role of control functions, and what we need finally are exactly  $n_k$  and  $B_k$ . The quantity  $n_k$  can be called a *crossover index* and  $B_k$ , *crossover amplitude*. With the found  $n_1$  and  $B_1$ , we define, analogously to (23), the first-order self-similar approximant

$$e_1^*(g) \equiv p_1^* \left( g, \frac{1}{3}, A_0^3 \right)$$

for the ground-state energy, which writes

$$e_1^*(g) = (a_0^3 + A_0^3 g)^{1/3}. \quad (37)$$

Comparing the values of (37) with numerical results [55] that can be treated as exact, we see that the maximal error, for  $g \geq 0$ , of Eq. (37) is  $-6.8\%$  occurring at  $g \approx 0.7$ .

Following the same way, we find the second-order self-similar crossover approximant

$$e_2^*(g) = \left[ \left( a_0^{9/2} + Cg \right)^{4/3} + B_2 g^2 \right]^{1/6}, \quad (38)$$

where two first terms in the asymptotic expansion (36) are used, and  $C = 0.1971$ ,  $B_2 = A_0^6 = 0.0888$ . The maximal error of (38) is  $-2.9\%$  at  $g \approx 0.3$ . Continuing the procedure, we get the third-order approximant

$$e_3^*(g) = \left\{ \left[ \left( a_0^{81/14} + C_1 g \right)^{4/3} + C_2 g^2 \right]^{7/6} + B_3 g^3 \right\}^{1/9}, \quad (39)$$

with  $C_1 = 0.1116$ ,  $C_2 = 0.0784$ ,  $B_3 = A_0^9 = 0.02648$ . The maximal error of (39) is  $-1.7\%$  at  $g \approx 0.1$ . In the fourth order, we obtain

$$e_4^*(g) = \left\{ \left\{ \left[ \left( a_0^{243/35} + D_1 g \right)^{4/3} + D_2 g^2 \right]^{7/6} + D_3 g^3 \right\}^{10/9} + B_4 g^4 \right\}^{1/12}, \quad (40)$$

where  $D_1 = 0.0625$ ,  $D_2 = 0.05005$ ,  $D_3 = 0.0131$ , and  $B_4 = A_0^{12} = 0.00789$ . The maximal error of (40) is  $-1.3\%$  at  $g \approx 0.1$ . The sequence is uniformly convergent, which can be seen from the monotonic decrease of errors from about  $7\%$  to  $1\%$ .

We would like to emphasize that our aim in the present paper is to suggest a *systematic analytical* method permitting one to derive explicit expressions describing crossover phenomena. The advantage of having accurate analytical formulas, as compared to numerical results of numerical methods, is in the simplicity of analyzing such formulas with respect to the variation of parameters entering these formulas. Also, having an analytical formula corresponding to a measurable quantity often gives more information about the studied system than just numbers. As an example, we may mention the geometric spectral inversion in quantum mechanics [57,58].

### III. FRÖHLICH POLARON

The Fröhlich optical polaron problem [3] is an interesting physical example of a crossover about which there existed a controversy lasting for around 30 years. Some researchers, analysing the polaron ground-state energy  $e(\alpha)$  as a function of the electron-phonon coupling parameter  $\alpha$ , found an indication to a phase transition from a state of freely moving weak-coupling polaron to a localized state of a strong-coupling polaron (see discussion in [15]). One of the



first such indications has been suggested by Gross [59]. However, as modern investigations show [4], there is no phase transition in the polaron problem but the latter is an example of a classical crossover between the weak-coupling and strong-coupling limits.

In the weak-coupling limit, the ground state of polaron has an asymptotic behaviour

$$e(\alpha) \simeq a_1\alpha + a_2\alpha^2 + a_3\alpha^3, \quad (\alpha \rightarrow 0), \quad (41)$$

with three well-established terms [15,60,61], in which

$$a_1 = -1, \quad a_2 = -1.591962 \times 10^{-2}, \quad a_3 = -0.806070 \times 10^{-3}.$$

In the strong-coupling limit, Miyake [62,63] obtained

$$e(\alpha) \simeq A_0\alpha^2 + A_2 + A_4\alpha^{-2} \quad (\alpha \rightarrow \infty), \quad (42)$$

where

$$A_0 = -0.108513, \quad A_2 = -2.836, \quad A_4 = -4.864.$$

The terms of the weak-coupling expansion are known here with a better precision than those of the strong-coupling expansion. In addition, the coefficients,  $a_k$  decrease as  $k$  increases, while  $A_k$  increase with  $k$ . Therefore, here we have to construct self-similar approximations from the right to the left, that is, starting from the perturbative expression (42), we find a self-similar approximant  $e_k^*(\alpha)$ , with control functions defined from the asymptotic condition

$$e_k^*(\alpha) \simeq e_{as}(\alpha) \quad (\alpha \rightarrow 0),$$

in which  $e_{as}(\alpha)$  is given by (41). The accuracy of the found self-similar approximants  $e_k^*(\alpha)$  can be evaluated by comparing them with the values  $e(\alpha)$  obtained by Monte Carlo numerical calculations [4,64]. As usual, the accuracy of self-similar crossover approximants is the worst in an intermediate region, where a weak-coupling expansion is sewed with a strong-coupling one. For the polaron energy  $e_k^*(\alpha)$ , the maximal error occurs at  $\alpha \approx 10$ .

The first-order crossover approximation gives

$$e_1^*(\alpha) = -\alpha (1 + B\alpha^2)^{1/2}, \quad (43)$$

with  $B = A_0^2 = 0.011775$ . The maximal error of (43) is  $-10.5\%$ . The second-order approximant is

$$e_2^*(\alpha) = -\alpha \left[ 1 + \alpha (B_0 + B_2\alpha^2)^{3/2} \right]^{1/4}, \quad (44)$$

where  $B_0 = 0.159468$ ,  $B_2 = A_0^{8/3} = 0.002679$ . The maximal error of (44) is  $-4.54\%$ . Finally, we may find the third-order self-similar approximant taking account of all three known terms in (41). This yields

$$e_3^*(\alpha) = -\alpha \left\{ 1 + \alpha \left[ C_0 + \alpha (C_1 + C_2\alpha^2)^{3/2} \right]^{5/4} \right\}^{1/6}, \quad (45)$$

where  $C_0 = 0.152804$ ,  $C_1 = 0.049617$ , and  $C_2 = A_0^{16/5} = 0.000819$ . The maximal error of (45) is  $-1.5\%$ . Again, we see that, with increasing approximation order, the accuracy of the found crossover approximants improves, from an error of about 10% to that of about 1%. The very simple formula (45) gives the same accuracy as the Feynman variational calculations [3,65].

#### IV. KONDO EFFECT

One of the most remarkable examples of crossover phenomena is given by the Kondo effect [2]. The behavior of the system, consisting of a local impurity spin and conduction electrons, interacting by means of an antiferromagnetic exchange of strength  $J$ , changes from asymptotically free at high temperatures to that of the impurity screened by electronic lump, via the crossover region whose onset is characterized by the Kondo temperature estimated as  $T_k = D \exp(-1/2J)$ , where  $D$  stands for the Fermi-energy of electrons. We consider below only the case of a single-channel Kondo model with the impurity local moment equal to  $1/2$ . Most of our knowledge about the problem comes from the exact Bethe ansatz solution [66,67], from the field-theoretical renormalization group (RG) [68-70], and from

the Wilson numerical renormalization group [71]. It was pointed out in Ref.[72], that the Bethe-ansatz solution cannot be extrapolated beyond the coupling constant  $J$  of order one (see also [73]). Field-theoretical RG results are valid only at  $J < 1$  as well. On the other hand, the strong-coupling limit,  $J \rightarrow \infty$ , of the Kondo model was considered in Ref.[72]. Only the numerical RG treatment of the Kondo problem is valid, formally, for arbitrary  $J$ . We suggest below a simple analytical approach valid for arbitrary  $J$ .

Within the framework of the field-theoretical RG in its application to the Kondo crossover, the central role is played by the so-called invariant charge or effective electron-electron coupling  $J_{inv}$  [68-70], measuring the intensity of electron-electron interactions via the impurity spin. The field-theoretical Gell-Mann-Low  $\beta$ -function, could be defined using the perturbation theory in the weak-coupling limit [68-70],

$$\beta(J) \simeq -2J^2 + 2J^3, \quad (J \ll 1), \quad (46)$$

or by means of a sophisticated bosonization technique in the strong-coupling limit [72]:

$$\beta(J) \simeq -c, \quad c \approx 0.377, \quad (J \rightarrow \infty). \quad (47)$$

The left crossover approximation, satisfying by design both known limits, can be obtained giving an improved, self-similarly renormalized Gell-Mann-Low function:

$$\beta^*(J) = -2J^2 \left(1 + \frac{\tau}{2}J\right)^{-2}, \quad \tau \equiv \sqrt{\frac{8}{c}} = 4.607, \quad (48)$$

and  $J_{inv}$  is given [68-70] by the equation

$$\int_J^{J_{inv}} \frac{dg}{\beta^*(g)} = \ln\left(\frac{\omega}{D}\right).$$

The last integral can be calculated explicitly and the result may be presented in the form

$$\begin{aligned} \Phi(J_{inv}) &= \ln\left(\frac{\omega}{T_k}\right), \\ \Phi(z) &= \frac{1}{2z} - \frac{\tau}{2} \ln(z) - \frac{\tau^2}{8} z, \end{aligned} \quad (49)$$

where  $\omega$  stands for the typical external parameter of the problem (temperature, magnetic field) and  $T_k$  is the typical internal energy scale, or the Kondo temperature

$$T_k = DJ^{\tau/2} \exp\left(-\frac{1}{2J} + \frac{\tau^2}{8}J\right). \quad (50)$$

This expression for the Kondo temperature has the same form as the famous Wilson numerical RG result,

$$T_k = \tilde{D}(J)(2J)^{1/2} \exp\left[-\frac{1}{2J} + 1.5824(2J)\right],$$

where  $\tilde{D}(J)$  is known to have a power series expansion in  $J$  [71]. The origin of the linear correction in the exponential, can be traced, therefore, to the strong-coupling limit. To our knowledge, other analytical approaches, including the Bethe ansatz solution, cannot capture it. The effective interaction determined by Eq. (49), grows to infinity as  $\omega$  goes to zero in agreement with the numerical RG and strong-coupling limit [71,72,74].

## V. ONE-DIMENSIONAL ANTIFERROMAGNET

An extreme caution is needed when any kind of perturbative or non-perturbative approach is applied to the one-dimensional Heisenberg antiferromagnet of arbitrary spin  $S$ . Even such a general method as Bethe ansatz fails for  $S > 1/2$ . Nevertheless, the crossover approach can be of use in this situation.

### A. Autocorrelation Function.

The Bethe ansatz, despite its failure in the general case of  $S > 1/2$ , allows one to find the magnetic properties of the Heisenberg antiferromagnetic (AF) spin chains of arbitrary spin, when a maximum of two deviations is allowed from the completely aligned (ferromagnetic) state [75]. The magnetization curve and pair correlations had been obtained explicitly for a strong magnetic field, close to the spin-flop transition. The expression for the autocorrelation function  $\Theta_0 = \langle S_0^z S_0^z \rangle$ , as a function of the number  $N$  of spins  $S$ , has a very simple and transparent form. In Ref. [75], an equivalent quantity

$$F_0(N) = \frac{\Theta_0}{S^2} - \left( \frac{S_T^z}{NS} \right)^2$$

is presented (and compared to numerical data) as a function of a parameter  $\sigma$  (demagnetization),

$$\sigma \equiv 1 - \frac{S_T^z}{NS}.$$

Here  $S_T^z$  stands for the magnetization of a spin-flop phase and is controlled by the magnetic field  $h$ , so that close to the saturation field  $h_s = 4S$  [75] one has:

$$\frac{S_T^z}{NS} = 1 - \frac{2}{\pi S} \left( 1 - \frac{h}{h_s} \right)^{1/2}.$$

Finally,  $F_0$  is presented in the following form:

$$F_0(\sigma) \simeq \frac{\sigma}{S} - \sigma^2 + \frac{1}{8} \pi^2 S^2 (2S - 1)^2 \sigma^4 \quad (\sigma \ll 1). \quad (51)$$

The last term in (51), proportional to  $(2S - 1)^2$ , clearly distinguishes an extra-contribution from the so-called  $C$ -states, typical for  $S \geq 1$  and absent for  $S = 1/2$ , with high probability of having two spin deviations on the same site [75]. An exact, independent on  $N$ , value

$$F_0 = \frac{2}{3}, \quad (\sigma = 1, S = 1),$$

is known too [75], and can be used as an asymptotic condition. Let us continue the expansion (51) from the region of small  $\sigma$  to the region of  $\sigma \sim 1$ , along the stable trajectory, ending at  $\sigma = 1$  at the value  $F_0 = \frac{2}{3}$ . In order to extend the validity of (51) for  $S \geq 1$ , let us add to (51) one more trial term  $\sim -\sigma^6$ , and find the corresponding effective time  $\tau$  from the crossover condition at the boundary point. The self-similar bootstrap procedure leads to the following crossover approximations :

$$F_0^*(\sigma) = \frac{\sigma}{S(1 + \sigma S)}, \quad S = \frac{1}{2}, \quad (52)$$

$$F_0^*(\sigma) = \frac{\sigma}{S} \exp \left[ -\sigma S \exp \left( -\frac{1}{8} \pi^2 S^2 (2S - 1)^2 \sigma^2 \exp \left( -\frac{A \sigma^2}{S^2 (2S - 1)^2} \right) \right) \right], \quad S \geq 1, \quad (53)$$

where  $A = 8\tau/\pi^2 = 0.253$ . At  $S = 1$ ,  $F_0^*(\sigma)$  agrees, both qualitatively and quantitatively, with the data of Fig. 3 from Ref. [75], the maximal error being  $\approx 4\%$ . The behavior of  $F_0^*(\sigma)$  for  $S = 1/2$  and  $S = 1$  is qualitatively similar (universal regime) only as  $\sigma \rightarrow 0$  ( $h \rightarrow h_s$ ), where  $C$ -states are suppressed by a magnetic field, and is different for all finite  $\sigma$  ( $h < h_s$ ) due to the contribution from  $C$ -states (non-universal regime). An onset of the regime dominated by  $C$ -states may be related to the inflection point of the curve  $F_0^*(\sigma)$  for  $\sigma \sim 0.5$ , emerging for  $S = 1$ , and absent for  $S = 1/2$ .

Consider the case of  $h \rightarrow h_s$ ,  $S = 1$ . Upon rapid (instant) switching off magnetic field down to the value  $h = 0$ , the correlations between spins should change from the behavior typical of the universal regime to that of the non-universal regime, dominated by  $C$ -states. One may suspect, therefore, that the typical time interval of relaxation of physical properties, such as the pair correlation function of spins and the magnetization, would be radically different for  $S = 1/2$  and  $S = 1$ . To be more specific, consider another physical property  $\Gamma$ , an effective interaction of two spin flips, which can be presented in the vicinity of the saturation field as a function of  $S$  and  $h_s - h$  [76]:

$$\Gamma(S, h) \sim \frac{2}{\sqrt{2}(\pi S \delta)^{-1} + 1 - 1/S}, \quad \delta = \sqrt{4 - \frac{h}{S}}. \quad (54)$$

As  $h \rightarrow h_s$ ,  $\Gamma$  remains positive for arbitrary spins and the system enters the universal regime, when all equilibrium physical properties for arbitrary spins could be derived from the "Bose-gas with repulsion" model [76-78]. As  $h$  is instantly set to zero, the magnetization should readjust itself from the values near to the saturation to the zero magnetization. The expression (54) for  $\Gamma$  can still be used in this situation as an estimate of the effective interaction of two spin flips, at least at the initial stage of relaxation. Then, from (54), it follows that  $\Gamma(S = 1/2, h = 0) \approx -3.64$ , i.e. acquires the negative sign different from that of  $\Gamma(S = 1, h = 0) \approx 8.89$ . The function  $\Gamma(S, h = 0)$  has a peak at  $S = 1$ , then saturates as  $S \rightarrow \infty$  to the positive value 2. Negative sign of  $\Gamma(S = 1/2)$  means an attraction of two spin flips and rapid collapse of the magnetized state to the state without magnetization, while positive  $\Gamma(S = 1)$  means repulsion and much longer relaxation time for the magnetization. I.e., we can expect an *anomalously slow relaxation* of the magnetization for spin 1, after instant switching off magnetic field from the value close to saturation down to zero, compared to the case of spin 1/2.

## B. Ground-State Energy

Spin-wave theory gives for the ground state energy  $E$  of the Heisenberg AF in the one-dimensional case the expansion in powers of inverse spin  $1/S$  (see [79] and Refs. therein):

$$E \simeq -S^2 \left(1 + \frac{\gamma}{2S}\right), \quad \gamma \approx 0.7. \quad (55)$$

The self-similarly renormalized expression, following from (55), is

$$E^* = -S^2 \exp\left(\frac{\gamma}{2S}\tau\right), \quad (56)$$

and at  $\tau = 1$ ,  $E^*(S = 1) = -1.419$ , approximating the "exact" numerical result  $-1.401$  [80] with an accuracy of 1.285%, which is an improvement as compared to the error  $-3.64\%$  of expression (55), corresponding to "bare" spin waves. For  $S = 2$ ,  $E^* = -4.765$ , in excellent agreement with the "exact" numerical result  $-4.761$  [81].

The error, calculated for the renormalized expression (56) for  $S = 1/2$ , is equal to 13.54%, becoming much worse than  $-4.06\%$  for the bare spin waves, as compared to the exact value  $E_0 = -0.44315$  [30]. An attempt to improve the result for  $S = 1$ , choosing the effective time  $\tau$  from the exact result at  $S = 1/2$ , gives the error of  $-5\%$  as  $S = 1$ , suggesting, that between  $S = 1/2$  and  $S = 1$ , some new physical mechanism enters the play, invalidating our attempt to match smoothly the ground state energies for quantum spins based only on renormalization of spin-wave formula. On the other hand, a successful estimation of the ground state energy for  $S = 1, 2$ , based on  $1/S$ -expansion, suggests that a similar mechanism works for all  $S \geq 1$  and quite an accurate estimate can be obtained from formula (56).

Motivated by the existence of exact results for the autocorrelation function, assume that the ground state energy could be expanded around  $E_0$  in powers of  $(S - \frac{1}{2})$ , i.e. introduce a trial  $(S - \frac{1}{2})$ -expansion around the exact solution at  $S = \frac{1}{2}$ :

$$E \sim - \left[ |E_0| + A \left( S - \frac{1}{2} \right) \right], \quad S \rightarrow \frac{1}{2}. \quad (57)$$

The coefficient  $A$  will be determined by matching Eq. (57) with the expression for the ground state energy  $E$ , as  $S \rightarrow \infty$ :

$$E \sim -S^2 \quad (S \rightarrow \infty). \quad (58)$$

Following the standard prescriptions of Section II, we obtain the left crossover approximation

$$E^* = - \left[ \sqrt{|E_0|} + \left( S - \frac{1}{2} \right) \right]^2, \quad A = 2|E_0|^{-1/2}, \quad (59)$$

with  $E^*(S = 1) = -1.359$ , approximating the exact result with the percentage error of  $-3\%$ , being only slightly better than the spin-wave result. In order to check the idea about similar mechanisms, forming the ground state energy for  $S \geq 1$ , rewrite (59) in the form

$$E^* = - \left[ S^2 + \left( 2\sqrt{|E_0|} - 1 \right) S + \left( |E_0| - \sqrt{|E_0|} + 1/4 \right) \right], \quad (60)$$

and consider (60) as another form of  $1/S$ -expansion. Applying to Eq. (60) the procedure of self-similar renormalization, we obtain:

$$E^{**} = -S^2 \exp \left( \frac{2\sqrt{|E_0|} - 1}{S} \right), \quad (61)$$

with  $E^{**}(S=1) = -1.393$ , and the error  $-0.57\%$ . For  $S=2$ ,  $E^{**} = -4.72$ , and the error is equal to  $-0.86\%$ . We again conclude, that for the ground state energy a simple crossover formula exists, covering the region from large spins to the small quantum spin  $S=1$ .

### C. Haldane Gap

Haldane [5] conjectured the existence of radically different elementary excitation spectra for arbitrary integer and half-odd-integer  $1d$  Heisenberg spins, the former case being gapped with the smallest value of the gap  $\Delta$  at  $k=\pi$ , while the latter case is gapless. In the limit of large  $S$ , Haldane used an approximate mapping onto the  $O(3)$  non-linear sigma-model, leading to the following behavior of the gap [5]:

$$\Delta \sim S^2 \exp(-\pi S), \quad S \rightarrow \infty. \quad (62)$$

Strictly speaking, formula (62) describes only the "slow" part of the full dependence and does not take into account the "fast" part, describing the gap oscillations with changing spin, with zeros at half-odd-integer spins and maxima at integer values. Nowadays, it is established beyond the reasonable doubt [82], that for the half-odd-integer spins

$$\Delta \equiv 0, \quad \left( S = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots \right). \quad (63)$$

For small integer spins  $S=1, 2$ , the values of the gap are known from an extensive numerical calculations. We suggest below a simple way to estimate  $\Delta$  for arbitrary integer spins, based on the self-similar renormalization of a trial  $(S - \frac{1}{2})$ -expansion for the Haldane gap and on the knowledge of the asymptotic form (62), as  $S \rightarrow \infty$ , together with the demand for the absence of the gap for half-odd-integer spins.

Let us write the trial expansion for the gap in the vicinity of the point  $S = \frac{1}{2}$  in the following form, satisfying the condition  $\Delta(S = \frac{1}{2}) = 0$ :

$$\Delta \sim a_2 \left( S - \frac{1}{2} \right)^2 + a_3 \left( S - \frac{1}{2} \right)^3 + a_4 \left( S - \frac{1}{2} \right)^4 + a_5 \left( S - \frac{1}{2} \right)^5 + \dots \quad \left( S \rightarrow \frac{1}{2} \right), \quad (64)$$

where  $a_k$  are positive. Following the general prescriptions of Section II, one can self-similarly renormalize (64) to the form

$$\Delta^* = a_2 \left( S - \frac{1}{2} \right)^2 \left[ 1 - C_1 \left( S - \frac{1}{2} \right) \right]^{n_1} \exp \left\{ C_2 \left( S - \frac{1}{2} \right)^2 \left[ 1 - C_3 \left( S - \frac{1}{2} \right) \right]^{n_2} \right\}. \quad (65)$$

We require that (65) agrees with (62), as  $S \rightarrow \infty$ . Demand also, that at  $S = 3/2$ ,  $\Delta = 0$ . Choosing the unknown coefficients and powers in (65) so that to satisfy the required conditions, we may come to

$$\Delta^* = \left( S - \frac{1}{2} \right)^2 \exp \left[ - \frac{(S - \frac{1}{2})^2}{S - \frac{3}{2}} \right], \quad (66)$$

where the value at  $S = \frac{3}{2}$  is defined as the limit from the right,  $S \rightarrow \frac{3}{2} + 0$ . At  $S=1$ ,  $\Delta^* = 0.412$ , in good agreement with the "exact" numerical value 0.4105 [80]; at  $S=2$ ,  $\Delta^* = 0.025$ , agreeing by the order of magnitude with numerical value 0.085(5) [81] or, even in better agreement with the value 0.05, quoted in Ref. [83]. Formula (66) can be generalized requiring that the exponential in Eq. (66) should have the form of an expansion in  $1/S$  as  $S \rightarrow \infty$  and zeros at  $S = \frac{5}{2}, \frac{7}{2}, \frac{9}{2}, \dots$ , which yields

$$\Delta^* = \left( S - \frac{1}{2} \right)^2 \exp \left[ - \left( \frac{(S - \frac{1}{2})^2}{(S - \frac{3}{2})} + \frac{(S - \frac{3}{2})^3}{(S - \frac{5}{2})^3} + \frac{(S - \frac{5}{2})^4}{(S - \frac{7}{2})^5} + \frac{(S - \frac{7}{2})^5}{(S - \frac{9}{2})^7} + \dots \right) \right], \quad (67)$$

where again the values at  $S = (2n+1)/2$  are defined as the limits from the right. The value of the gap at  $S=1$ , given by Eq. (67), remains practically the same as above, while at  $S=2$ ,  $\Delta^* = 0.068$ . The gap, when described by formula (67), practically vanishes for all integer  $S \geq 3$ , in agreement with the conclusion of Ref. [81].

## D. Other Characteristics

Self-similar approximants can be constructed for other characteristics as well. Here we briefly mention only a couple of examples. Staggered magnetization  $\Sigma$  of the antiferromagnetic anisotropic Ising–Heisenberg model of spin-1/2, as a function of the anisotropy parameter  $\gamma$  (equal to zero for the Ising and one for the Heisenberg model), can be presented as the expansion, valid at small  $\gamma$  [84,85]:

$$\Sigma(\gamma) \simeq 1 - \gamma^2 - \frac{1}{4}\gamma^4 - \dots \quad (68)$$

As  $\gamma = 1$ , according to Ref. [85], the long-range order parameter  $\omega_\infty = \Sigma^2(\gamma)$  should disappear, i.e.  $\Sigma(1) = 0$ . Let us continue expression (68), from the region of  $\gamma \ll 1$ , to the region of  $\gamma \sim 1$ , satisfying the boundary condition for the disappearance of the long-range order. Then the left crossover approximation

$$\Sigma^*(\gamma) = (1 - \gamma^2) \exp\left(-\frac{1}{4}\gamma^4\right) \quad (69)$$

satisfies the right boundary condition. Comparing  $\omega_\infty^* = (\Sigma^*(\gamma))^2$ , with the extrapolation of numerical data, presented in Fig.30 of Ref. [85], we found that they almost coincide.

At zero temperature, the dispersion, known from the linear spin-wave theory, is modified by the factor  $Z$  in the spin-wave velocity, and the expansion for  $Z$  in powers of the inverse coordination number  $1/z$  was obtained [86,87]:

$$Z \simeq 1 + \frac{1}{4Sz} + \frac{3}{16Sz^2} + \dots,$$

We continue this expression from the small values of  $1/z$  to arbitrary  $z$ , while the value of spin is fixed, and determine the effective time  $\tau$  from the exact value of  $Z = \pi/2$ , at  $S = 1/2$ ,  $z = 2$  [87]. The left crossover approximation has the following form:

$$Z^* = 1 + \frac{1}{4Sz} \exp\left(\frac{3}{4z}\tau\right), \quad \tau = 2.21. \quad (70)$$

At  $S = 1/2$ ,  $z = 4$  (square lattice) we obtain from Eq. (70)  $Z^* = 1.189$ , in excellent agreement with the results obtained by different methods [87]. At  $S = 1/2$ ,  $z = 6$ , the case corresponding to a simple cubic lattice,  $Z = 1.11$ , i.e the quantum corrections to the spin velocity remain important.

## VI. COLLECTIVE EXCITATIONS

The knowledge of the elementary excitation spectrum is one of the key points for the description of many-body problems. Dealing with this extremely complicated problem, one often encounters the situation when the elementary excitation spectrum  $\omega(k)$  is known for two different regions of wave vector  $k$ . In the hydrodynamic region,  $k \rightarrow 0$ , the form  $\omega(k)$  could be determined either from experiment or theoretically. In the short-wave region,  $k \rightarrow \infty$ , a dispersion corresponding to free particles should recover. Using the self-similar renormalization, it is possible to reconstruct  $\omega(k)$  for arbitrary  $k$ . Consider some problems of this kind, frequently occurring in condensed matter physics.

### A. Bogolubov Spectrum

The case of a linear in  $k$  spectrum, as  $k \rightarrow 0$ , and of quasi-free massive particles, as  $k \rightarrow \infty$ , is of the most general type when Bose systems are considered. This kind of behavior is inherent to Bose systems and does not depend on the details of an interaction potential [6]. Consider the case of an anomalous sound dispersion, corresponding to an instability of the spectrum, as  $k$  increases [7]. The following asymptotic expressions are available:

$$\begin{aligned} \omega(k) &\simeq ck(1 + \gamma k^2), \quad \gamma > 0 \quad (k \rightarrow 0), \\ \omega(k) &\simeq \frac{k^2}{2m^*} \quad (k \rightarrow \infty). \end{aligned} \quad (71)$$

Here  $c$  is the velocity of sound,  $\gamma$  is responsible for the instability of the spectrum and  $m^*$  is an effective mass. The left crossover approximation can be derived following the standard prescriptions of Section II, which gives the result identical to the Bogolubov spectrum of a weakly non-ideal Bose gas:

$$\omega^*(k) = ck\sqrt{1 + \left(\frac{k}{2m^*c}\right)^2}. \quad (72)$$

Note, that in distinction from the microscopic Bogolubov approach, valid for a diluted Bose system, formula (72) may be used for arbitrary densities, assuming that the parameters  $c$  and  $m^*$  are taken from experiment. It looks rather intriguing that the same formula (72), that is usually derived with some lengthy calculations, can be immediately obtained by self-similarly interpolating the simple asymptotic expressions (71).

## B. Liquid Helium Spectrum

Consider the case when more terms in the hydrodynamic limit are available, but the "anomalous" dispersion coefficient  $\gamma$  is very small. Also, free particles are replaced by quasi-free "dressed" particles with an effective mass  $m^*$ . The asymptotic behaviour of the spectrum is as follows:

$$\begin{aligned} \omega(k) &\simeq ck(1 + \gamma k^2 - \delta k^4), \quad \gamma \approx 0, \quad \delta > 0 \quad (k \rightarrow 0), \\ \omega(k) &= \frac{k^2}{2m^*} \quad (k \gg 1\text{\AA}^{-1}). \end{aligned} \quad (73)$$

This situation is typical for liquid  $He^4$ , where  $\gamma = 0 \pm 0.05\text{\AA}^2$ ,  $\delta = 0.29 \pm 0.03\text{\AA}^4$  [88], and  $m^* = 2 - 3m(He^4)$  [89]. The crossover approximant derived from (73) reads

$$\omega^*(k) = ck \left[ (\exp(-\delta k^4))^6 + \left(\frac{k}{2m^*c}\right)^6 \right]^{1/6}. \quad (74)$$

Expression (74) generalizes the Bogolubov spectrum (72). The main difference originates from the region of the intermediate  $k \sim 1$ . Formula (74) describes the experimental data for the elementary excitation spectrum of liquid  $He^4$  [88] both qualitatively, predicting the existence of roton minimum even for the "bare" mass  $m^* = m(He^4)$ , and quantitatively, with the maximal percentage error of about 20%. The value of the effective mass  $m^* = 2 - 3m(He^4)$  may have some relation to the formation of two-particle and three-particle correlated states [90]. We took above for estimates the value of the sound velocity equal to  $2.4 \cdot 10^4 \text{cm/s}$ .

Our approach to deriving the spectrum for arbitrary  $k$  corresponds to the Feynman approach [65], when only the information about the short-wave and long-wave parts of the structure factor  $S(k)$  are used. Then, instead of the phenomenological Feynman formula  $\omega(k) = k^2/2mS(k)$ , we apply the self-similar renormalization. The result is a Bogolubov-type formula. Thus, a bridge between the Bogolubov and Feynman approaches to the spectrum of Bose systems [91] is established. Formula (74) is better qualitatively than the original Bogolubov spectrum (71), since it predicts the maxon-roton region, and better quantitatively than Feynman formulae, especially in the roton region. Here, the Feynman formula works with an error of about 100%, while Eq. (74), in the worst case, gives an error about 10%. Our formula (74), is a three-parametric representation of the spectrum of liquid  $He^4$ , with parameters  $c$ ,  $\delta$  and  $m^*$  coming from the regions of long, intermediate and short-wave lengths, respectively.

The case of a stable sound-like spectrum, as  $k \rightarrow 0$ , and of quasi-free particles, as  $k \rightarrow \infty$ , can also correspond to a collective-excitation branch in liquid  $He^3$  [92]. The following asymptotic expressions are available:

$$\begin{aligned} \omega(k) &\simeq ck(1 - |\gamma|k^2), \quad (k \rightarrow 0), \\ \omega(k) &\simeq \frac{k^2}{2m^*}, \quad (k \rightarrow \infty). \end{aligned} \quad (75)$$

By analogy with the case of  $He^4$ , we find the spectrum

$$\omega^*(k) = ck \left[ (\exp(-|\gamma|k^2))^4 + \left(\frac{k}{2m^*c}\right)^4 \right]^{1/4}. \quad (76)$$

However, since phonons in liquid  $He^3$  are intertwined, in the region of intermediate wave vectors, with other collective excitations, it is impossible to observe rotons.

### C. Spectrum with Gap

Assume that the spectrum has a gap, as  $k \rightarrow 0$ , and possesses a minimum at this point, while, as  $k \rightarrow \infty$ , it becomes linear:

$$\begin{aligned}\omega(k) &\simeq \Delta + \alpha k^2, \quad \alpha > 0 \quad (k \rightarrow 0), \\ \omega(k) &\simeq vk, \quad (k \rightarrow \infty).\end{aligned}\tag{77}$$

The left crossover approximation can be readily obtained, leading to the expression

$$\omega^*(k) = \Delta \sqrt{1 + \left(\frac{vk}{\Delta}\right)^2},\tag{78}$$

analogous to the spectrum of the Bardeen–Cooper–Schrieffer model of superconductivity.

### D. Dynamical Scaling

The characteristic frequency  $\omega_c(\zeta, k)$ , appearing in the dynamical scaling hypothesis [8,93] and proportional to an inverse characteristic relaxation time of an order parameter, has two asymptotic forms, depending on the ratio  $k/\zeta$ , where  $\zeta$  stands for the inverse correlation length. We shall discuss below only the behavior of density–density correlations in liquid systems [8]. Asymptotic expansions in the hydrodynamic regime ( $k/\zeta \ll 1$ ) and in the fluctuation regime ( $k/\zeta \gg 1$ ) are known:

$$\begin{aligned}\omega_c(\zeta, k) &\simeq D_T k^2 \left[ 1 + B \left(\frac{k}{\zeta}\right)^2 + \dots \right], \quad B > 0, \quad \left(\frac{k}{\zeta} \ll 1\right), \\ \omega_c(\zeta, k) &\simeq A k^z \left[ 1 + A' \left(\frac{\zeta}{k}\right)^2 + \dots \right], \quad \left(\frac{k}{\zeta} \gg 1\right),\end{aligned}\tag{79}$$

where  $D_T$  is a thermal diffusivity and  $z$  is the dynamical critical index, which cannot be determined self-consistently within the framework of the dynamical scaling. The value of  $B$  is estimated as  $B = 1$  [94], or  $B = 3/5$  [95].

Assume that the values of  $z$  and  $A$  are known. Then one can reconstruct the analytical expression for the characteristic frequency for arbitrary  $k/\zeta$  obtaining the following left crossover approximation

$$\omega_c^*(\zeta, k) = D_T k^2 \left[ 1 + C \left(\frac{k}{\zeta}\right)^2 \right]^n,\tag{80}$$

where

$$C = \zeta^2 \left(\frac{A}{D_T}\right)^{2/(z-2)}, \quad n = \frac{z}{2} - 1.$$

For  $z = 3$  [8], we obtain  $n = 1/2$ . If now we plug into expression (80) the dependencies of  $D_T \sim \epsilon^{\gamma-\alpha}$  and  $\zeta \sim \epsilon^\nu$  on the distance  $\epsilon$  from the critical point [8], then we recover immediately the well-known relation between the critical indices  $z$ ,  $\gamma$ ,  $\nu$  and  $\alpha$ , that is,  $z = 2 + (\gamma - \alpha)/\nu$  (all definitions are standard and may be found in Ref. [8]), which represents one of the central results of the dynamical scaling hypothesis. From this scaling relation, the dynamical critical index could be estimated from the values of three other indices.

## VII. CONCLUSION

We suggested a general approach to describing crossover phenomena of arbitrary nature. The approach permits one to construct an accurate approximation for a function in the whole domain of its variable from asymptotic expansions near the boundaries. The minimal information needed for obtaining a self-similar interpolation formula is two terms of an expansion near one of the boundaries and the limiting value at another boundary. Having only three such terms,



it is already possible to get a reasonable approximation for the sought function in the total crossover region. When more terms are available, the procedure may be continued improving the accuracy of approximations. An important feature of the method is that the self-similar crossover approximants always preserve the correct structure of the asymptotic expansions at both boundaries of the interpolation region. This is a clear advantage of the self-similar approach as compared to often used heuristic interpolations that may spoil the structure of the asymptotic expansions.

The possibility of obtaining accurate approximations from an extremely scarce information, when no other methods work, is based on the following three points: (i) The idea of self-similar renormalization group treating the transfer from one approximation to another as the evolution of a dynamical system, approximation cascade. (ii) The requirement that this evolution be invariant with respect to algebraic transformations. (iii) The use of control functions providing the stability and convergence of procedure.

Control functions introduced under the algebraic self-similar renormalization play, for the crossover problem, the role of effective crossover indices and effective crossover times. Depending on whether we start the renormalization procedure from an expansion either near the left or near the right boundary, we may distinguish the left and right crossover indices and, respectively, the left and right crossover times. Similarly, the resulting expressions for the sought function may be called the left and the right crossover approximations.

The form of the resulting self-similar approximations depends on the properties of the asymptotic expansions used. Mathematically equivalent expansions lead to the same form of crossover approximations. For example, compare the ground state energy of the Fröhlich polaron as a function of the coupling parameter and the spectrum of collective excitations as a function of the wave vector. The weak coupling series in powers of the coupling parameter is analogous to the long-wave spectrum in powers of the wave vector. The strong coupling limit for the optic polaron is similar to the short-wave limit for the collective spectrum. As a result of this, the crossover approximation for the polaron energy has the same dependence on the coupling parameter as the crossover approximation for the collective spectrum on the wave vector. Thus, physically different quantities may have the same mathematical representation as functions of the corresponding variables. Keeping this in mind, we may say that there exist the *classes of universality* of crossover phenomena.

It is worth emphasizing that the crossover approximations derived by applying the approach developed usually combine good accuracy with simplicity. This suggests that the self-similar renormalization provides a natural tool for extracting the maximal information from very short perturbative series that are impossible to analyze by other methods. Moreover, this makes us to think that self-similarity, in some sense, is hidden in asymptotic series. This is why the self-similar renormalization becomes a natural effective tool of extracting such a hidden information. Different physical examples presented in this paper prove as well that this is also a general tool applicable to arbitrary crossover phenomena.

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